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LETTER TO THE EDITOR

Shifted $1/N$ expansion and scalar potential in the Dirac equation

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Abstract. The shifted $1/N$ expansion method has been applied to a linear scalar potential in the Dirac equation to obtain the relativistic energy eigenvalues. The results are compared with those obtained by Rein and by Gunion and Li.

In recent years the shifted $1/N$ expansion method has been used by several authors (Sukhatme *et al* 1983, Roy and Roychoudhury 1987, Dutt *et al* 1986a, b) to determine the energy eigenvalues of the Schrödinger equation for some important potentials. So far, however the use of this method has been restricted to non-relativistic problems. Some time ago Miramontes and Pajares (1984) studied the large-N limit of relativistic equations but their result was only applied to the Coulomb problem which is an exactly solvable case.

In the present letter a formalism has been developed to determine the eigenvalues of the Dirac equation for radially symmetric scalar potentials assuming the largeness of rest energy compared with the relativistic corrections. Though the formalism applies for any radially symmetric scalar potential, numerical results are obtained for the one proportional to r. The confining potential for quarks being predominantly Lorentz scalar, a scalar potential of the form $U=\lambda r$ has been used by several authors (Rein 1977, Gunion and Li 1975, Critchfield 1976) to explain the J/ψ spectrum and hence accurate numerical values exist for comparison with our results.

Mathematically the scalar potential is easier to treat because if we keep only first-order relativistic corrections then the effective potential can easily be treated by the usual non-relativistic $1/N$ expansion method. The Dirac equation for a scalar potential $U(r)$ can be written in the following N-dimensional form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{(N_j-2)(N_j-2+2s)}{4r^2}-[(m+U)^2-W^2]\right)G=\frac{\mathrm{d}U}{\mathrm{d}r}F \tag{1}$$

and

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{(N_j-2)(N_j-2-2s)}{4r^2}-[(m+U)^2-W^2]\right)F=\frac{\mathrm{d}U}{\mathrm{d}r}G \tag{2}$$

where we have used the Dirac spinors (we have taken $\hbar=c=1$)

$$\psi_{js}^{M}=\begin{cases}(1/r)F(r)\,Y_{lj}^{M} & l=j+\frac{1}{2}s \qquad s=\pm 1\\ (1/r)\mathrm{i}G(r)\,Y_{l'j}^{M} & l'=j-\frac{1}{2}s\end{cases} \tag{3}$$

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and

$$N_j = N + 2j. \tag{4}$$

Equations (1) and (2) can be written in the form

$$(W + m + U)F = \frac{dG}{dr} + \frac{\chi}{r} G \tag{5}$$

$$(-W + m + U)G = \frac{dF}{dr} - \frac{\chi}{r} F \tag{6}$$

where

$$\chi = \tfrac{1}{2} s(N_j - 2). \tag{7}$$

When

$$s = +1 \qquad N = 3 \qquad \chi = l \tag{8a}$$

and when

$$s = -1 \qquad N = 3 \qquad \chi = -(l+1) \tag{8b}$$

then (5) and (6) reduce to the usual equations for a scalar potential.

Now eliminating F and writing $W = E + m$, we get from (5) and (6)

$$\frac{d^2G}{dr^2} - \frac{\chi(\chi+1)}{r^2} G = (U^2 + 2mU - E^2 - 2Em)G + \frac{1}{2m + E + U} \frac{dU}{dr} \left(\frac{dG}{dr} + \frac{\chi}{r} G \right). \tag{9}$$

For $U(r)$ of the form $U(r) = \lambda r$, (9) can be simplified further by expanding $1/(2m + E + U)$ in the form $(1/2m)\,[1 - (E + U)/2m + \ldots]$. The contribution by the term proportional to $(E + U)/4m^2$ being of higher order can be evaluated as a correction after the leading-order calculation has been done following the usual $1/N$ expansion method. Neglecting terms proportional to $(E + U)/4m^2$ we have, from (9),

$$\frac{d^2\phi}{dr^2} - \frac{(k-1)(k-3)}{4r^2} \phi(r) + [e - V(r)]\phi(r) = 0 \tag{10}$$

where

$$e = E^2 + 2Em - \lambda^2/16m^2 \tag{10a}$$

$$V(r) = \lambda^2 r^2 + 2\lambda m r + \chi\lambda/2mr \tag{10b}$$

and

$$\phi(r) = \exp\left(\frac{-r}{4m} \frac{dV}{dr} \right) G(r) \tag{10c}$$

where in analogy with the usual $1/N$ expansion method

$$k = N + 2l. \tag{10d}$$

Equation (10) easily yields to the usual non-relativistic $1/N$ shifted expansion method and since this has been discussed in detail by Sukhatme *et al* (1984) we only quote the results here. The radial equation (10) is written in terms of the shifted

variable $\bar{k}=k-a$ and then the leading contribution to the energy comes from the effective potential

$$e=\bar{k}^2 V_{\text{eff}}(r)=\frac{\bar{k}^2}{r_2^0}\left(\frac{1}{4}+\frac{r_0^2 V(r_0)}{Q}\right) \tag{11}$$

where r_0 is so chosen that $\partial V_{\text{eff}}/\partial r=0$ at $r=r_0$, which leads to

$$2r_0^2 V'(r_0)=Q \tag{12}$$

where $Q=\bar{k}^2$. Note that we have kept the term $\chi\lambda/2mr$ in the effective potential though its contribution is much less than the rest of the effective potential. This was done because one can then follow the non-relativisitic $1/N$ expansion method without any change in the formalism and since we are interested in terms only up to $O(\lambda/m^2)$, this prescription is correct up to that order.

The next contribution to the energy term is

$$\frac{\bar{k}}{r_0^2}[(n+\tfrac{1}{2})\omega-\tfrac{1}{2}(2-a)].$$

The shift a is so chosen as to make this correction vanish. This choice of a gives the correct energy value for non-relativistic Coulomb and harmonic oscillator potentials as pointed out by Imbo *et al* (1984). Here ω is given by

$$\omega=\left(3+r_0\frac{V''(r_0)}{V(r_0)}\right)^{1/2}. \tag{13}$$

Higher-order corrections have been given elsewhere (see, for example, Imbo *et al* 1984). In the following we give a collection of formulae required to calculate the energy eigenvalues. r_0 is given by the root of the equation

$$N+2l-2+(2n_r+1)\left(3+\frac{r_0V''(r_0)}{V'(r_0)}\right)^{1/2}=(2r_0^3V'(r_0))^{1/2}. \tag{13a}$$

The energy e is given by an expansion in powers of $1/\bar{k}$, where $\bar{k}=N+2l-a$

$$e_{n,l}=\frac{\bar{k}}{r_0^2}\left[\frac{h^2\bar{k}}{4}+\frac{r_0^2\bar{k}V(r_0)}{Q}+\frac{\beta^{(1)}}{\bar{k}}+\frac{\beta^{(2)}}{\bar{k}^2}+O\left(\frac{1}{\bar{k}^3}\right)\right] \tag{14}$$

where

$$\begin{aligned}\beta^{(1)}&=\tfrac{1}{4}(1-a)(3-a)+(1+2n_r)\tilde{\varepsilon}_2+3(1+2n_r+2n_r^2)\tilde{\varepsilon}_4\\&\quad-\omega^{-1}[\tilde{\varepsilon}_1^2+6(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_3+(11+30n_r+30n_r^2)\tilde{\varepsilon}_3^2]\end{aligned} \tag{15}$$

$$\begin{aligned}\beta^{(2)}&=(1+2n_r)\tilde{\delta}_2+3(1+2n_r+2n_r^2)\tilde{\delta}_4+5(3+8n_r+6n_r^2+4n_r^3)\tilde{\delta}_6\\&\quad-\omega^{-1}[(1+2n_r)\tilde{\varepsilon}_2^2+12(1+2n_r+2n_r^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_4\\&\quad+2(21+59n_r+51n_r^2+34n_r^3)\tilde{\varepsilon}_4^2+2\tilde{\varepsilon}_1\tilde{\delta}_1+6(1+2n_r)\tilde{\varepsilon}_1\tilde{\delta}_3\\&\quad+30(1+2n_r+2n_r^2)\tilde{\varepsilon}_1\tilde{\delta}_5+6(1+2n_r)\tilde{\varepsilon}_3\tilde{\delta}_1\\&\quad+2(11+30n_r+30n_r^2)\tilde{\varepsilon}_3\tilde{\delta}_3+10(13+40n_r+42n_r^2+28n_r^3)\tilde{\varepsilon}_3\tilde{\delta}_5]\\&\quad+\omega^{-2}[4\tilde{\varepsilon}_1^2\tilde{\varepsilon}_2+36(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_2\tilde{\varepsilon}_3+8(11+30n_r+30n_r^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_3^2\\&\quad+24(1+2n_r)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_4+8(31+78n_r+78n_r^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3\tilde{\varepsilon}_4\\&\quad+12(57+189n_r+225n_r^2+150n_r^3)\tilde{\varepsilon}_3^2\tilde{\varepsilon}_4]\\&\quad-\omega^{-3}[8\tilde{\varepsilon}_1^3\tilde{\varepsilon}_3+108(1+2n_r)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_3^2+48(11+30n_r+30n_r^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3^3\\&\quad+30(31+109n_r+141n_r^2+94n_r^3)\tilde{\varepsilon}_3^4]\end{aligned} \tag{16}$$

in which

$$\tilde{\varepsilon}_j = \varepsilon_j/\omega^{j/2} \qquad \tilde{\delta}_j = \delta_j/\omega^{j/2}$$

$$\delta_1 = -\tfrac{2}{3}\delta_2 = -\tfrac{1}{2}(1-a)(3-a)$$

$$\delta_3 = -\tfrac{4}{5}\delta_4 = 2\varepsilon_1 = -\tfrac{4}{3}\varepsilon_2 = 2(2-a)$$

$$\varepsilon_3 = -1 + \frac{r_0^5 V'''(r_0)}{6Q} \qquad \varepsilon_4 = \tfrac{5}{4} + \frac{r_0^6 V''''(r_0)}{24Q}$$

$$\delta_5 = -\tfrac{3}{2} + \frac{r_0^7 V'''''(r_0)}{120Q} \qquad \delta_6 = \tfrac{7}{4} + \frac{r_0^8 V''''''(r_0)}{720Q}.$$

From e the eigenvalue E is then calculated from the relation (10a). We list our energy eigenvalues for various energy states and from some values of l in tables 1-3 and compare them with numerical results obtained by Rein (table 1) and by Gunion and Li for the energy levels of the J/ψ spectrum for a linear potential (tables 2 and 3).

It is clear from tables 1-3 that our results are in very good agreement with the numerical results given by Rein (1977) and Gunion and Li (1975). As is expected the agreement is best when the value of $m/\sqrt{\lambda}$ is the highest. Thus the shifted $1/N$ expansion works well for a Dirac scalar potential of the form $V(r) = \lambda r$.

Table 1. Energy eigenvalues (in GeV) of the ground state and two excited s states as well as the first p state. The values given in parentheses are those given by Rein (1977). W is the total energy, $W = E + m$.

$m/\sqrt{\lambda}$	$W_0/\sqrt{\lambda}$	$W^p/\sqrt{\lambda}$	$W_1/\sqrt{\lambda}$	$W_2/\sqrt{\lambda}$
1	2.334	3.049	3.283	3.962
	(2.40)	(3.00)	(3.32)	(3.99)
4	5.081	5.556	5.846	6.437
	(5.08)	(5.58)	(5.85)	(6.44)
6	6.973	7.398	7.673	8.222
	(6.98)	(7.40)	(7.67)	(8.22)

Table 2. Dirac results for part of the J/ψ spectrum calculated after a model of Gunion and Li (1975) with $\lambda = 0.137\ \text{GeV}^2$, $m = 1.12$ GeV and $j = l + \frac{1}{2}$. The values in parentheses are those given by Gunion and Li. All energy values are in GeV.

	l			
n_r	0	1	2	3
0	3.106	3.449	3.733	3.982
	(3.103)	(3.442)	(3.725)	(3.973)
1	3.698	3.948	4.175	4.383
	(3.7)	(3.946)	(4.170)	(4.377)
2	4.152	4.359	4.553	4.736
	(4.158)	(4.36)	(4.551)	(4.732)
3	4.535	4.717	4.889	5.054
	(4.545)	(4.72)	(4.89)	(5.053)
4	4.874	5.038	5.195	5.345
	(4.886)	(5.043)	(5.198)	(5.346)

Table 3. Dirac results for part of the J/ψ spectrum after Gunion and Li (1975) with $\lambda = 0.137\ \text{GeV}^2$ $m = 1.12\ \text{GeV}$ and $j = l - \frac{1}{2}$. The values in parentheses are those obtained by Gunion and Li (1975). All energy values are in GeV.

	l			
n_r	1	2	3	4
0	3.471	3.760	4.010	4.236
	(3.47)	(3.757)	(4.006)	(4.23)
1	3.954	4.194	4.406	4.600
	(3.965)	(4.194)	(4.403)	(4.597)
2	4.370	4.568	4.754	4.920
	(4.374)	(4.560)	(4.753)	(4.926)
3	4.726	4.902	5.069	5.228
	(4.731)	(4.95)	(5.07)	—
4	5.045	5.205	5.359	5.507
	(5.053)	(5.21)	(5.361)	(5.506)

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